TECHNICAL NOTE

Adiabatic compressible flow in parallel ducts: an approximate but rapid method of solution

G. J. Parker
Department of Mechanical Engineering, University of Canterbury, Christchurch, New Zealand

By examining the nature of exact solutions for adiabatic one-dimensional flow of perfect gases in ducts, we have developed an approximate method of solution which is direct and, hence, much faster while still being acceptably accurate.

Keywords: adiabatic compressible flow; approximate solution

Introduction

The analysis of one-dimensional compressible (Fanno-type) flow is well documented (e.g., Ref. 1). A key equation, deriving from the energy and momentum equations, is

$$\frac{4fl}{d}=\frac{1}{\gamma}\left[\frac{1}{M_1^2}-\frac{1}{M_2^2}\right]-\frac{\gamma+1}{2\gamma}\ln\left[\frac{(\gamma-1)M_1^2+2}{(\gamma-1)M_2^2+2}\right] \qquad (1)$$

where M_1 and M_2 are the Mach numbers at inlet to and outlet from the parallel duct. This equation has to be solved in conjunction with equations for property ratios (e.g., p_1/p_2), expressed in terms of M_1 and M_2, and the continuity equation to give the mass flow rate w $(=\rho_1 A_1 u_1)$.

A common problem is to determine the mass flow rate through a given duct for a known pressure difference across it. The solution requires an iterative process, in effect determining a pair of values for M_1 and M_2 which satisfy all the equations. The usual method adopted is to plot or tabulate from Equation 1 values of $4fl/d$ for various M_1 with $M_2=1$ (i.e., choked flow with sonic conditions at the downstream end). This gives the maximum value of $4fl/d$ for that value of M_1.

If the pressure ratio between 1 and 2 is the given parameter, the procedure is to select pairs of values of M_1 and M_2 so that

$$\left[\frac{4fl}{d}\right]_{\text{actual}}=\left[\frac{4fl}{d}\right]_{\substack{\max\\ M=M_1}}-\left[\frac{4fl}{d}\right]_{\substack{\max\\ M=M_2}} \qquad (2)$$

and

$$\frac{p_2}{p_2}=\left[\frac{M_1}{M_2}\right]\left[\frac{(\gamma-1)M_1^2+2}{(\gamma-1)M_2^2+2}\right]^{1/2} \qquad (3)$$

It is also not uncommon for the flow into the parallel duct to be through a convergent nozzle at inlet, so the above pressure ratio term can be expanded to incorporate isentropic flow through the nozzle; thus

$$\frac{p_2}{p_0}=\left[\frac{M_1}{M_2}\right]\left[\frac{(\gamma-1)M_1^2+2}{(\gamma-1)M_2^2+2}\right]^{1/2}\left[\frac{2}{(\gamma-1)M_1^2+2}\right]^{\gamma/(\gamma-1)} \qquad (4)$$

where p_0 is the stagnation pressure and p_2/p_0 would be the given pressure ratio.

Address reprint requests to Dr. Parker at the Department of Mechanical Engineering, University of Canterbury, Christchurch, New Zealand.

Received 7 March 1988; accepted for publication 15 August 1988

The iterative solution is not difficult and can be readily programmed for computer solution. However, if this solution is just part of a much larger program (e.g., compressible flow through the complex passages in pop safety valves[2]), then this step may be an iterative step within one or more other iterative steps, and a faster more direct procedure becomes worth seeking, to save computation time.

Exact solution

Results from computations described above are conveniently shown in dimensionless form, plotting a dimensionless mass flow rate W $(=w/A(p_0\rho_0)^{1/2})$ against the pressure ratio r $(=p_2/p_0)$ for various values of $4fl/d$. A typical result for $\gamma=1.4$ is shown in Figure 1.

Observations from exact solutions

Several interesting features can be noted from Figure 1, and from similar diagrams for other values of γ. First, the locus of the maximum values of W at choked conditions is linear with

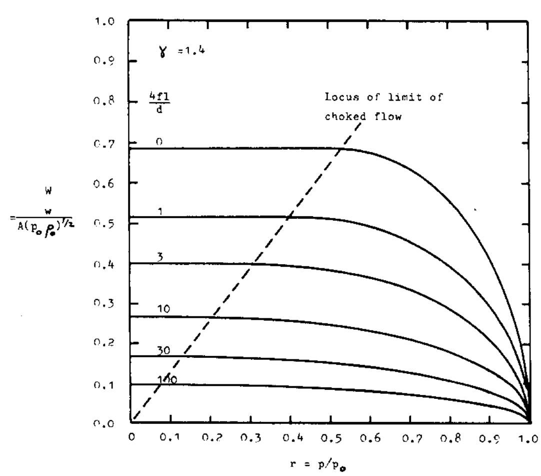

Figure 1 Results from exact analysis for parallel duct with inlet nozzle, showing dimensionless flow rate against overall pressure ratio for various values of $4fl/d$

the choked pressure ratio, as shown by the dotted line in the figure. (This can also be deduced analytically.) Second, if the unchoked regions of the curves are plotted on a relative basis as W_{rel} against r_{rel}, then the curves collapse to follow closely a single curve, as shown in Figure 2. Furthermore, the curve approximates very closely to a quadrant of a circle with the equation

$$W_{rel}^2 = 1 - r_{rel}^2 \tag{5}$$

Third, the effect of variations in γ can be reduced by including γ in $4fl/d$. The curves represented in Figure 2 have already been plotted using the parameter $\gamma 4fl/d$. Last, a relatively simple functional relation can be used to express the choked dimensionless mass flow rate in terms of $\gamma 4fl/d$. If expressed as a ratio to the nozzle choked flow (i.e., for $\gamma 4fl/d = 0$), then the following expression represents a good fit to the exact solutions:

$$\frac{W_{cd}}{W_{cn}} = \left[1 + a\left[\frac{\gamma 4fl}{d}\right]^b\right]^c \tag{6}$$

where a, b, and c are constants.

Procedure for approximate solution

To calculate the mass flow rate through a given duct (plus inlet nozzle) for a known overall pressure ratio, we do the following:

(1) Compute the value of $\gamma 4fl/d$. f is strictly not a constant, being dependent on the Reynolds number and roughness ratio, but $f = 0.005$ is a value which serves most purposes with acceptable accuracy.

(2) From standard isentropic relations, determine the choked mass flow rate and choked pressure ratio for the nozzle alone.

$$W_{cn} = \left[\gamma\left[\frac{2}{\gamma+1}\right]^{(\gamma+1)/(\gamma-1)}\right]^{1/2}$$
$$r_{cn} = \left[\frac{2}{\gamma+1}\right]^{\gamma/(\gamma-1)} \tag{7}$$

These values have to be determined only once, at the beginning of the calculations, since they are functions of γ only.

(3) For the given value of $\gamma 4fl/d$, determine the dimensionless mass flow ratio W_{cd}/W_{cn} using the functional relation Equation 6.

(4) Because of the linear relationship, the ratio of pressure ratios r_{cd}/r_{cn} equals the mass flow ratio obtained in step (3).

(5) Then obtain the pressure ratio r_{cd}.

(6) Obtain the relative pressure ratio for the required pressure ratio r: $r_{rel} = (r - r_{cd})/(1 - r_{cd})$.

(7) Then obtain the relative flow rate from W_{rel} $(= W/W_{cd}) = (1 - r_{rel}^2)^{1/2}$.

(8) The dimensionless mass flow rate is then, from steps (2), (3), and (7), $W = W_{rel}(W_{cd}/W_{cn})W_{cn}$.

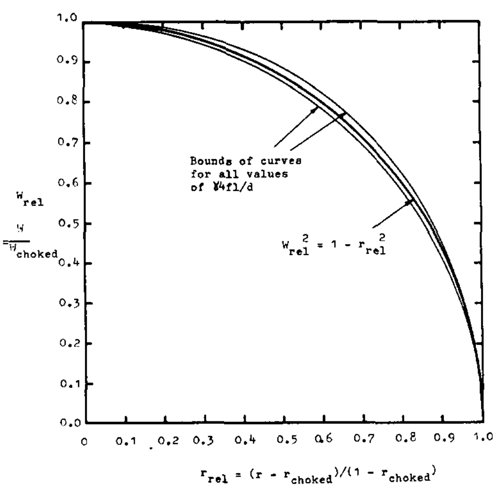

Figure 2 Plot of relative dimensionless mass flow against relative pressure ratio, showing that curves for all values of $\gamma 4fl/d$ fit closely to a quadrant of a circle

Comparison with exact solutions

This procedure has been compared with the solutions from exact analyses for a range of γ from 1.1 to 1.7, a range of $\gamma 4fl/d$ from 0 to 10,000, and a range of pressure ratios from 1.0 to the appropriate choked pressure ratio ($r_{rel} = 0$). Values for the constants used in Equation 6 were $a = 0.4386$, $b = 0.7966$, $c = -0.6211$. The errors are plotted in Figures 3(a), (b), and (c).

It can be seen that, for choked conditions ($r_{rel} = 0$) the error is small (less than $\pm 1\%$), except for high values of $\gamma 4fl/d$ (which would normally correspond to very long ducts). For pressure ratios in the midrange ($r_{rel} = 0.5$) the error ranges from -3% at $\gamma 4fl/d = 0.3$ to $+3\%$ at $\gamma 4fl/d = 10{,}000$. When the pressure ratio is close to unity (i.e., very small flow rates), the error is still within $\pm 3\%$, except for the values of $\gamma 4fl/d$ below about

Notation

a	Constant in Equation 6
A	Duct cross-sectional area, m^2
b	Constant in Equation 6
c	Constant in Equation 6
d	Duct diameter, m
f	Friction factor (defined from head loss $= 4flu^2/2gd$)
l	Duct length, m
M	Mach number
p	Pressure, Pa
r	Pressure ratio p_2/p_0
r_{cn}	Pressure ratio for choked conditions at nozzle discharge
r_{cd}	Pressure ratio for choked conditions at duct discharge
r_{rel}	Relative pressure ratio $= (r - r_{cd})/(1 - r_{cd})$
u	Gas velocity, m/s
w	Mass rate of flow, kg/s
W	Dimensionless mass rate of flow $= w/A(p_0\rho_0)^{1/2}$
W_{cn}	Dimensionless choked flow rate for nozzle alone
W_{cd}	Dimensionless choked flow rate for duct
W_{rel}	Relative dimensionless flow rate $= W/W_{cd}$
γ	Isentropic expansion index (p/ρ^γ = const.) = ratio of specific heats for a perfect gas
ρ	Density, kg/m^3

Subscripts

0	Stagnation condition
1	Duct inlet
2	Duct outlet

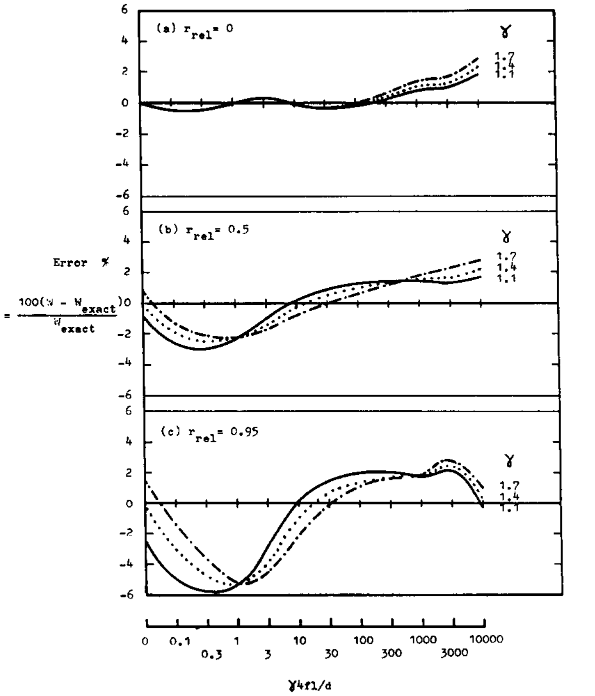

Figure 3 Error in mass flow estimation by using the approximate solution for a range of values of $\gamma 4fl/d$ and γ

6.0, when the error can reach up to −6%. The effect of variation in the value of γ is also seen to be small, except for the low values of $\gamma 4fl/d$ at pressure ratios near unity.

Conclusion

The error in flow rate incurred by using the approximate method would normally be acceptable, considering the uncertainty associated with the value of friction factor and the assumption of one-dimensional flow. For the type of problem for which it was developed, the fact that it is 100 times faster in the computer than the iterative method makes it a most acceptable method.

References

1 Shapiro, A. H. *The Dynamics and Thermodynamics of Compressible Fluid Flow*, Vol. 1, Ronald Press, New York, 1953
2 Parker, G. J. "Pop" safety valves: a compressible flow analysis. *Int. J. Heat and Fluid Flow*, 1985, **6**(4), 279–283